Gravity between non-point masses

A gravitational force is a force between exactly two masses. It is calculated using Newton's gravitational equation

$$G = \gamma \cdot \frac{m_0 \cdot m_1}{a^2} \tag{1}$$

$$\text{with } \gamma = 6{,}6726 \cdot 10^{-11} \ \frac{Nm^2}{kg^2}$$

$$Gravitational\ constant$$

For *a* in this equation is used the distance of the centers of mass in both masses. This means that for the calculation of gravity it is assumed that the masses of m_0 and m_1 (Fig. 1) are concentrated in the centers of mass and are not distributed in a spatial area *V*. However, this assumption only applies with sufficient accuracy if the dimensions of the masses are small compared to their distance *a*. If the distance is reduced, however, this becomes inaccurate. For the calculation of gravity, the forces of particular mass elements dm_0 and dm_1 must be summed up, the gravitational forces of which cannot be regarded as the same due to the different distances. The mass elements with a smaller distance will require larger gravitational forces, those with a larger distance smaller ones. The summed gravity of all mass elements of both masses then gives the total gravity of both masses. For the calculation of the total gravity, a fictitious point *G* will be created in each of the masses as the concentration point of the total mass (Fig. 1), which is not identical to the center of mass *S* of the masses. The point should be called the *gravitational point of the non-point mass*. There is a deviation of the gravitational point from the center of mass. The distance of these points *G* of both masses, a_g, is then to be inserted as the distance of the masses in Newton's gravitational equation (1). The aim of the following calculation is to determine the deviation $d = a - a_g$ of the point *G*. As the method for calculating the deviation, a factor *f* is to be determined (deviation factor), whose product with *a* gives the equivalent distance a_g with that the gravitational force by the equation (1) can be calculated:

$$a \cdot f = a_g$$

The deviation results from this as

$$d = a - a_g = a \cdot (1 - f) \tag{2}$$

The following simplified conditions are assumed for the calculation:

- Both masses m_0 and m_1 are homogeneous with constant mass density δ.
- The dimension of the mass m_0 is negligibly small compared to the distance *a*, so that it can be treated as a point mass.
- The mass m_1 has the shape of a sphere with the radius *r*.

The sphere with the mass m_1 is cut along the *x*-coordinate in circular disks of thickness *dx* perpendicular to the x-direction, the cross-sectional area *s* of which is limited by the sphere surface. When determining the gravitation between m_0 and the mass elements dm_s on a circular disc, force components in the y-direction arise due to the angle between the x- direction and the direction m_0 - dm_s. However, these components compensate each other because there is an equally large component in the opposite direction for each *y*-component of gravitation m_0 - dm_s. The mass of the disk can thus be viewed as if it were concentrated in the center of the disk.

It should be noted that this is an approximation. Exactly considered, for this procedure not flat disks should be used, but spherical shells, which are drawn with the distance size *x* in the spherical volume. This approach would result in a considerably greater computing effort than in the approximation. An alternative approach is the formation of the sum of all gravitational forces dm_s within a disc, which are calculated according to (1) with the hypotenuses of the triangles from the distance *x* and the associated *y*-value. These further calculations will be the subject of a subsequent work. In the following illustration, the mass of a disk of thickness *dx* should be regarded as the mass element dm_1 of the spherical mass m_1.

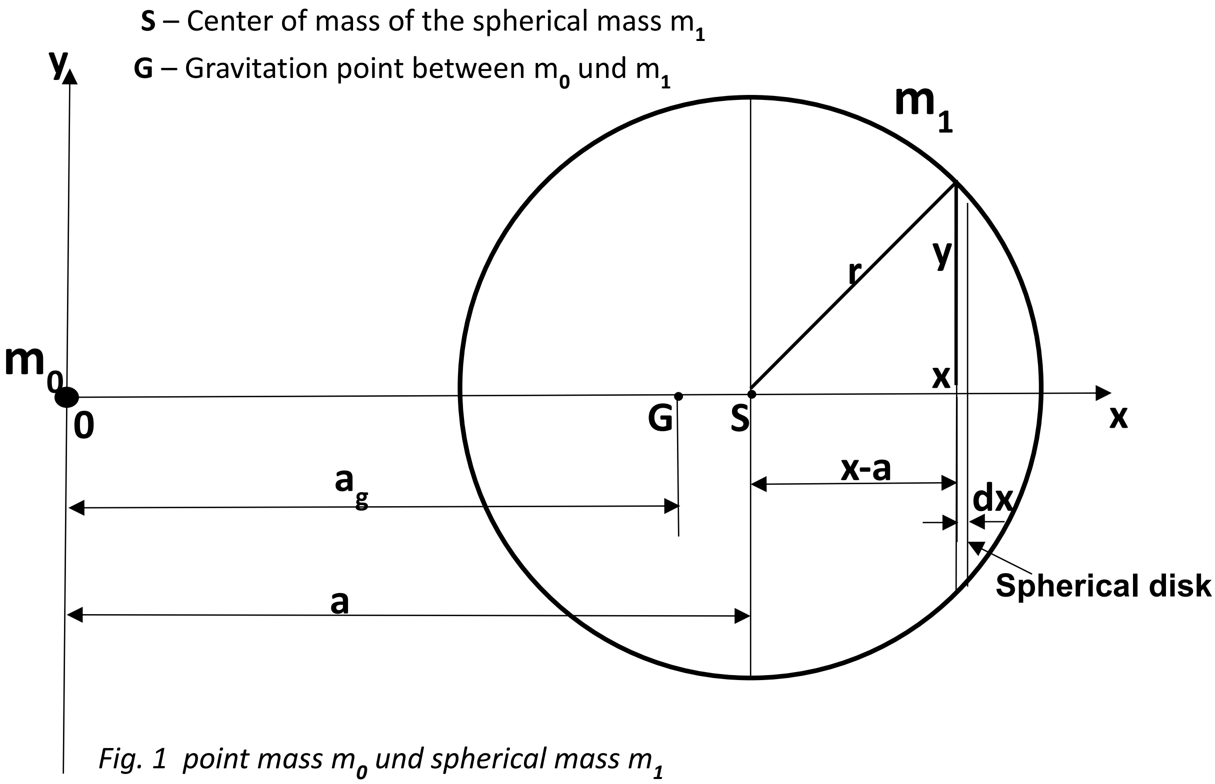

Fig. 1 point mass m_0 und spherical mass m_1

The cross-sectional area of a circular disc is

$$s = \pi \cdot y^2 \tag{3}$$

with

$$y^2 = r^2 - (x-a)^2 . \tag{4}$$

(3) and (4) after algebraic revision lead to

$$s = \pi(r^2 - a^2 + 2ax - x^2) . \tag{5}$$

As can be easily checked, the cross-sectional area *s* at the outer limits of the mass m_1, that is, for $x = a - r$ and for $x = a + r$, is zero. For $x = a$, arises $s = \pi \cdot r^2$ what was to be expected. Outside the mass m_1, for $x = a - r - \Delta$ and $x = a + r + \Delta$ where there is Δ any positive value, negative values for *s* arise which have no physical content.

For spherical disk is

$$dm_1 = \delta \cdot s \cdot dx \tag{6}$$

and with (5)

$$dm_1 = \delta \cdot \pi \cdot (r^2 - a^2 + 2ax - x^2) \cdot dx \qquad (7).$$

The solution of the integral $\int_{a-r}^{a+r} dm_1$ results $\delta \cdot \frac{4}{3}\pi \cdot r^3$, what was to be expected.

The prerequisites for integrating the gravitational force between m_0 and m_1 are thus present.

The total gravity over all dm_1 can be calculated. For this has to put to

$$dG = \gamma \cdot \frac{m_0 \cdot dm_1}{x^2} \cdot dx \tag{8}$$

and at long last with (7)

$$dG = \gamma \cdot \left(\frac{m_0 \cdot \delta \cdot \pi \cdot (r^2 - a^2 + 2ax - x^2)}{x^2} \right) \cdot dx$$

The integral over the limits of the sphere radius r has the form

$$G = \gamma \cdot m_0 \cdot \delta \cdot \pi \cdot \int_{a-r}^{a+r} \left(\frac{r^2 - a^2}{x^2} + \frac{2a}{x} - 1 \right) dx \tag{9}$$

with the solution

$$G = \gamma \cdot m_0 \cdot \delta \cdot \pi \cdot \left[\frac{a^2 - r^2}{x} + 2a \ln x - x \right]_{a-r}^{a+r} \tag{10}$$

After the integral limits are set, arises

$$G = \gamma \cdot m_0 \cdot \delta \cdot \pi \cdot \left[2a \cdot \ln \frac{a+r}{a-r} - 4r \right] \tag{11}$$

Equation (11) is the form of Newton's gravitational equation modified for non-point masses. The gravitational point of mass m_1 is not yet recognizable in this form. In order to make visible the deviation of the gravitational point, the equation must be transformed so that is created an invariant form to Newton's equation (1). For this is used the result of the integration of (6), which is

$$m_1 = \delta \cdot \pi \cdot \frac{4}{3} \cdot r^3$$

Equation (11) multiplied with

$$\frac{m_1}{\delta \cdot \frac{4}{3} \cdot \pi \cdot r^3} = 1$$

This leads to

$$G = \gamma \cdot m_0 \cdot m_1 \cdot \frac{3}{2r^2} \cdot \left[\frac{a}{r} \cdot \ln \frac{a+r}{a-r} - 2 \right]$$

After a further multiplication with

$$\frac{a^2}{a^2} = 1$$

you get

$$G = \gamma \cdot \frac{m_0 \cdot m_1}{a^2} \cdot \left[\frac{3}{2} \cdot \left(\frac{a}{r}\right)^3 \cdot \ln\frac{a+r}{a-r} - 3 \cdot \left(\frac{a}{r}\right)^2\right] \tag{12}$$

The square brackets contained the square of the reciprocal value of the deviation factor *f* sought at the beginning, with which the value of the deviation of the gravitational point can be calculated according to (2)

$$f^2 = \frac{1}{\frac{3}{2} \cdot \left(\frac{a}{r}\right)^3 \cdot \ln\frac{\frac{a}{r}+1}{\frac{a}{r}-1} - 3 \cdot \left(\frac{a}{r}\right)^2} \tag{13}$$

It can be seen here that the deviation factor *f* depends on the ratio of the distance to the radius of the mass m_1. If one calls this ratio *k*, one finally finds

$$f = \frac{1}{k \cdot \sqrt{3 \cdot \left(\frac{k}{2}\ln\frac{k+1}{k-1} - 1\right)}} \tag{14}$$

With large values of *k*, *f* approaches to 1:

$$\lim_{k\to\infty} f = 1\,.$$

This means that the deviation for large ratio values of the distance m_0 - m_1 to the radius of the mass m_1 becomes negligibly small, so that in equation (1) using the distance of the center of mass leads to very precise values for the gravitational force.

For small values of *k* there are non-negligible deviation values which have to be taken into consideration when calculating the gravitational force.

The limit case of the contact of the masses m_0 and m_1 (that is if $k = 1$) cannot be treated with the method used, it would lead to an infinite gravitation. This can be explained by the fact that the concentration of the masses either at the center of mass or at the gravitational point assumed for the calculation cannot exist in reality. Masses are not punctiform, they always occupy a spatial area *V*. The assumption $V = 0$ meant an infinite density. This would correspond to a singularity, the existence of which is controversial. In Table 1, which was developed with Excel, this is visible in the last lines of the selected areas, in which no values are shown.

The following table (Table 1) of the relationships between *a, r, k* and *f* shows the magnitude and the practical meaning of the deviation. It can be seen that under cosmic scales, in which *k* is usually much larger than 100, there is no need to take them into consideration. The deviation is negligible. In the vicinity of the mass m_1, the deviation takes on values that cannot be neglected.

In some sources, for example, methods for calculating the earth's mass can be found using the gravitational value of another mass near the earth, where the earth's radius is used to calculate this gravity, which means that the center of gravity is equated with the center of mass. This is not correct because a significant deviation must be taken into consideration for the mass near the earth (Table 2).

a	*r*	*k*	*f*	*a*	*r*	*k*	*f*	*a*	*r*	*k*	*f*
1000,00000	1,00000	1000,00000	1,00000	2,00000	1,00000	2,00000	0,91927	1,00100	1,00000	1,00100	0,34441
900,00000	1,00000	900,00000	1,00000	1,90000	1,00000	1,90000	0,90974	1,00090	1,00000	1,00090	0,34128
800,00000	1,00000	800,00000	1,00000	1,80000	1,00000	1,80000	0,89833	1,00080	1,00000	1,00080	0,33787
700,00000	1,00000	700,00000	1,00000	1,70000	1,00000	1,70000	0,88448	1,00070	1,00000	1,00070	0,33412
600,00000	1,00000	600,00000	1,00000	1,60000	1,00000	1,60000	0,86738	1,00060	1,00000	1,00060	0,32993
500,00000	1,00000	500,00000	1,00000	1,50000	1,00000	1,50000	0,84583	1,00050	1,00000	1,00050	0,32518
400,00000	1,00000	400,00000	1,00000	1,40000	1,00000	1,40000	0,81789	1,00040	1,00000	1,00040	0,31962
300,00000	1,00000	300,00000	1,00000	1,30000	1,00000	1,30000	0,78026	1,00030	1,00000	1,00030	0,31284
200,00000	1,00000	200,00000	0,99999	1,20000	1,00000	1,20000	0,72637	1,00020	1,00000	1,00020	0,30397
100,00000	1,00000	100,00000	0,99997	1,10000	1,00000	1,10000	0,63909	1,00010	1,00000	1,00010	0,29038
0,00000	1,00000	0,00000	#ZAHL!	1,00000	1,00000	1,00000	#DIV/0!	1,00000	1,00000	1,00000	#DIV/0!
100,00000	1,00000	100,00000	0,99997	1,10000	1,00000	1,10000	0,63909	1,00010	1,00000	1,00010	0,29038
90,00000	1,00000	90,00000	0,99996	1,09000	1,00000	1,09000	0,62681	1,00009	1,00000	1,00009	0,28847
80,00000	1,00000	80,00000	0,99995	1,08000	1,00000	1,08000	0,61346	1,00008	1,00000	1,00008	0,28638
70,00000	1,00000	70,00000	0,99994	1,07000	1,00000	1,07000	0,59882	1,00007	1,00000	1,00007	0,28406
60,00000	1,00000	60,00000	0,99992	1,06000	1,00000	1,06000	0,58256	1,00006	1,00000	1,00006	0,28145
50,00000	1,00000	50,00000	0,99988	1,05000	1,00000	1,05000	0,56425	1,00005	1,00000	1,00005	0,27845
40,00000	1,00000	40,00000	0,99981	1,04000	1,00000	1,04000	0,54318	1,00004	1,00000	1,00004	0,27491
30,00000	1,00000	30,00000	0,99967	1,03000	1,00000	1,03000	0,51810	1,00003	1,00000	1,00003	0,27054
20,00000	1,00000	20,00000	0,99925	1,02000	1,00000	1,02000	0,48649	1,00002	1,00000	1,00002	0,26472
10,00000	1,00000	10,00000	0,99699	1,01000	1,00000	1,01000	0,44127	1,00001	1,00000	1,00001	0,25557
0,00000	1,00000	0,00000	#ZAHL!	1,00000	1,00000	1,00000	#DIV/0!	1,00000	1,00000	1,00000	#DIV/0!
10,00000	1,00000	10,00000	0,99699	1,01000	1,00000	1,01000	0,44127	1,000010	1,00000	1,000010	0,25557
9,00000	1,00000	9,00000	0,99628	1,00900	1,00000	1,00900	0,43523	1,000009	1,00000	1,000009	0,25427
8,00000	1,00000	8,00000	0,99529	1,00800	1,00000	1,00800	0,42873	1,000008	1,00000	1,000008	0,25283
7,00000	1,00000	7,00000	0,99384	1,00700	1,00000	1,00700	0,42163	1,000007	1,00000	1,000007	0,25122
6,00000	1,00000	6,00000	0,99160	1,00600	1,00000	1,00600	0,41380	1,000006	1,00000	1,000006	0,24941
5,00000	1,00000	5,00000	0,98787	1,00500	1,00000	1,00500	0,40501	1,000005	1,00000	1,000005	0,24732
4,00000	1,00000	4,00000	0,98093	1,00400	1,00000	1,00400	0,39488	1,000004	1,00000	1,000004	0,24482
3,00000	1,00000	3,00000	0,96563	1,00300	1,00000	1,00300	0,38276	1,000003	1,00000	1,000003	0,24172
2,00000	1,00000	2,00000	0,91927	1,00200	1,00000	1,00200	0,36727	1,000002	1,00000	1,000002	0,23753
1,00000	1,00000	1,00000	#DIV/0!	1,00100	1,00000	1,00100	0,34441	1,000001	1,00000	1,000001	0,23086
0,00000	1,00000	0,00000	#ZAHL!	1,00000	1,00000	1,00000	#DIV/0!	1,000000	1,00000	1,000000	#DIV/0!

Table 1: Table representation of the deviation factor f as a function of k:

cosmic examples:					**deviation for a point mass near the earth**		
	diameter in km	**max**	**min**		***k***	**above earth**	**f**
sun	1.392.684	1.392.814	1.392.554		1,00015696	1 km	0,29901
earth	12.742	12.756	12.714		1,00007848	500 m	0,28604
moon	3.476	no oblateness			1,00003139	200 m	0,27122
	distance				1,00001570	100 m	0,26141
sun - erath	149.600.000	152.100.000	147.100.000		1,00000785	50 m	0,25259
earth - moon	371.500	384.000	359.000		1,00000314	20 m	0,24220
					1,00000157	10 m	0,23514
k		***k***			1,00000078	5 m	0,22866
29	earth - moon	107			1,00000031	2 m	0,22086
107	sun - earth	11.741			1,00000016	1 m	0,21546
					1,00000008	0,5 m	0,21044
					1,00000002	0,1 m	0,20002
					1,00000000	0 m	#DIV/0!

Table 2: Numerical examples in cosmic sizes and close to earth

The calculation presented here is a theoretical approach, which shows that the concentration of the masses in its centers of mass assumed for the calculation of gravitation is not generally valid, but must be considered depending on the ratio of the distance of the gravitating masses to their sizes. The homogeneity of the density of the masses is an ideal case that is extremely rare in the cosmos.

In practice, for example when calculating the gravitational values of the earth with nearby masses, further calculations have to be carried out since the density of the earth is not homogeneous. According to PREM (*Preliminary Reference Earth Model*), a reference model developed by Adam M. Dziewonski and Don L. Anderson in 1981 for seismic velocities, density, pressure and other physical parameters in the interior of the earth, the density of the earth in the inner core of the earth is by a factor up to 13 larger than in the outer layers (Fig. 2).

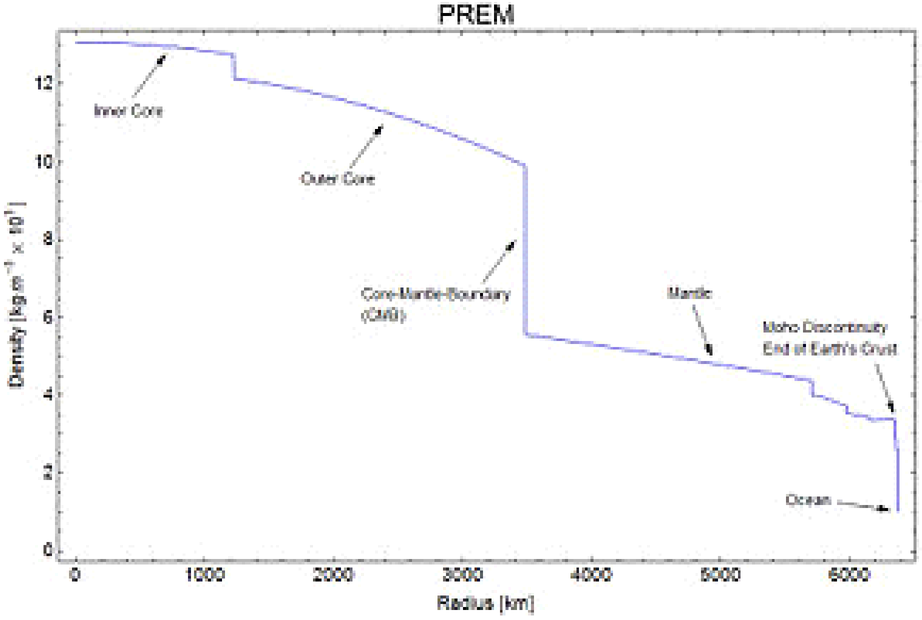

Fig. 2 Earth's density as a function of depth

This can be explained by the fact that in the process of the formation of the earth in its liquid phase, the heavy elements (iron, nickel) have moved into the center due to their internal gravitation (gravitational segregation). As expected, this leads to a reduction of the deviation, because according to this model, 33% of the total earth mass is contained in 17% of the internal earth volume.

The execution of the deviations calculations relevant for the earth is the subject of a subsequent work, for which goal-oriented research has to be carried out beforehand.

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